Olympiad Track

10 Questions from previous Maths Olympiads discussed

1. *Solve the system of equations for real x and y,*

$$5x\left(1+\frac{1}{x^2+y^2}\right)=12,\quad 5y\left(1-\frac{1}{x^2+y^2}\right)=4.$$

Sol. Given that : $5x\left(1+\frac{1}{x^2+y^2}\right)=12$

$$\therefore \quad 25x^2=\frac{144}{\left(1+\frac{1}{x^2+y^2}\right)^2} \quad \dots (1)$$

and similarly, we can write the second equation

$$25y^2=\frac{16}{\left(1-\frac{1}{x^2+y^2}\right)^2} \quad \dots (2)$$

On adding Eqs. (1) and (2), we get

$$25(x^2+y^2)=\frac{144}{\left(1+\frac{1}{x^2+y^2}\right)^2}+\frac{16}{\left(1-\frac{1}{x^2+y^2}\right)^2} \quad \dots (3)$$

Let $\frac{1}{x^2+y^2}=t$ so that $x^2+y^2=\frac{1}{t}$.

Now $$\frac{25}{t}=\frac{144}{(1+t)^2}+\frac{16}{(1-t)^2}$$

$$\Rightarrow \quad 144t\,(1-t)^2+16t\,(1+t)^2=25(1-t^2)^2$$

$$\Rightarrow \quad 32t\,(5t^2-8t+5)=25(t^4-2t^2+1)$$

On dividing both sides by t^2, we get

$$32\left\{5\left(t+\frac{1}{t}\right)-8\right\}=25\left\{\left(t+\frac{1}{t}\right)^2-4\right\}$$

On putting $t+\frac{1}{t}=\alpha$ in above equation, we get

$$25\alpha^2-160\alpha+156=0$$

$$\Rightarrow \quad t+\frac{1}{t}=\alpha=\frac{6}{5}\text{ or }\frac{26}{5}$$

$$\Rightarrow \quad 5t^2-6t+5=0\text{ or }5t^2-26t+5=0$$

Since, the discriminant of $5t^2-6t+5=0$ is $36-100<0$, there is no real root.

For $5t^2-26t+5=0$, the roots are 5 and $\frac{1}{5}$.

Thus, $x^2+y^2=\frac{1}{5}$ or $x^2+y^2=5$.

If $x^2+y^2=5$, then $5x\left(1+\frac{1}{5}\right)=12$ and $5y\left(1-\frac{1}{5}\right)=4$ and thus solving, we get

$$x=2\text{ and }y=1.$$

If $x^2+y^2=\frac{1}{5}$, then $5x\,(1+5)=12$ and $5y\,(1-5)=4$ and, thus solving we get

$$x=\frac{2}{5}\text{ and }y=-\frac{1}{5}$$

$\therefore$ Two solutions $x=2,\ y=1$ and

$$x=\frac{2}{5},\quad y=-\frac{1}{5}.$$

2. *Find all integers x, y satisfying*

$$(x-y)^2+2y^2=27.$$

Sol. $(x-y)^2,\ 2y^2\geq 0$ and since $2y^2$ is even, $(x-y)^2$ is an odd and hence $(x-y)$ should be odd. So the different possibilities for $(x-y)^2$ and y^2 are (1, 13) (9, 9) (25, 1). Corresponding to $y^2=13$, there is no solution because y is an integer. So, taking the other two pairs, we get

$$x-y=\pm 3,\ y=\pm 3 \quad \dots (1)$$

$$x-y=\pm 5,\ y=\pm 1 \quad \dots (2)$$

On solving Eq. (1), we get

$$(0,\ 3), (6,\ 3),\ (0,\ -3),\ (-6,\ -3)$$

On solving Eq. (2), we get

$$(6,\ 1),\ (-4,\ 1),\ (-6,\ -1),\ (4,\ -1).$$

3. *Determine $x,\ y,\ z\in R$, such that*

$$2x^2+y^2+2z^2-8x+2y-2xy+2xz-16z+35=0.$$

Sol. $2x^2+y^2+2z^2-8x+2y-2xy+2xz-16z+35=0$

$$\Rightarrow (x-y)^2+(x+z)^2+z^2-16z-8x+2y+35=0$$

$$\Rightarrow \quad (x-y-1)^2+(x+z-3)^2+z^2-10z+25=0$$

$$\Rightarrow \quad (x-y-1)^2+(x+z-3)^2+(z-5)^2=0$$

Thus, $x-y-1=0,\ x+z-3=0$ and $z=5$

$\therefore \quad x=-2,\ y=-3$ and $z=5$

Thus, the solution is $x=-2,\ y=-3$ and $z=5$.

4. *Solve the system :*

$$(x+y)(x+y+z)=18$$
$$(y+z)(x+y+z)=30$$
$$(z+x)(x+y+z)=2L$$

in terms of L.

Sol. On adding the three equations, we get

$$2(x+y+z)^2=48+2L$$

or $\quad x+y+z=\sqrt{24+L}$

On dividing the three equations by

$(x+y+z)=\sqrt{24+L}$, we get

$$x+y=\frac{18}{\sqrt{24+L}},\ y+z=\frac{30}{\sqrt{24+L}},$$

$$z+x=\frac{2L}{\sqrt{24+L}}$$

and solving, we get

$$x=\frac{(24+L)-30}{\sqrt{24+L}}=\frac{L-6}{\sqrt{24+L}}$$

$$y=\frac{(24+L)-2L}{\sqrt{24+L}}=\frac{24-L}{\sqrt{24+L}}$$

and $\quad z=\dfrac{24+L-18}{\sqrt{24+L}}=\dfrac{L+6}{\sqrt{24+L}}$.

5. *Find all pairs of integers x, y such that*

$$(xy-1)^2-(x+1)^2=(y+1)^2.$$

Sol. We have, $(xy-1)^2=(x+1)^2+(y+1)^2$

$$\Rightarrow \quad (xy-1)^2-(x+1)^2=(y+1)^2$$

$$\Rightarrow \quad (xy-x-2)(xy+x)=(y+1)^2$$

$$\Rightarrow \quad x(xy-x-2)(y+1)=(y+1)^2$$

$$\Rightarrow \quad (y+1)\{x(xy-x-2)-(y+1)\}=0$$

If $y=-1$, then x takes all values from the set of integers.

Similarly, we also get

$$(x+1)\{y(xy-y-2)-(x+1)\}=0$$

If $x=-1$, then y takes all values from the set of integers.

If $x\neq-1,\ y\neq-1$, we get

$$x(xy-x-2)(y+1)=(y+1)^2$$

$$\Rightarrow \quad x(xy-x-2)=y+1 \quad (\because\ y\neq-1)$$

$$\Rightarrow \quad x^2y-x^2-2x-y-1=0$$

$$\Rightarrow \quad y(x-1)(x+1)=(x+1)^2$$

$$\Rightarrow \quad y(x-1)=x+1 \quad (\because\ x\neq-1)$$

$$\Rightarrow \quad y=\frac{x+1}{x-1}, \text{ is an integer}$$

If $x=0$, then $y=-1$ and when $x=2$, then $y=3$, and similarly when $x=3$, then $y=2$ for all other values of x, y is not an integer.

Hence, solution set is

$(3,\ 2),\ (2,\ 3),\ (x,\ -1),\ (1,\ y)$;

where $x,\ y\in$ integer.

6. *Solve :*

$$xy+x+y=23$$
$$yz+y+z=31$$
$$zx+z+x=47$$

Sol. Given that,

$$xy+x+y=23 \qquad \text{... (1)}$$
$$yz+y+z=31 \qquad \text{... (2)}$$
$$zx+z+x=47 \qquad \text{... (3)}$$

On adding 1 to both sides of Eq. (1), we get

$$xy+x+y+1=24$$

$$\Rightarrow \quad (x+1)(y+1)=24 \qquad \text{... (4)}$$

Similarly, we get

$$(y+1)(z+1)=32 \qquad \text{... (5)}$$

and $\quad (z+1)(x+1)=48 \qquad \text{... (6)}$

On multiplying Eqs. (4), (5) and (6), we get

$$(x+1)^2(y+1)^2(z+1)^2=24\times32\times48$$
$$=24\times24\times64$$

$$\Rightarrow \quad (x+1)(y+1)(z+1)=\pm(24\times8)=\pm192$$

Since, none of $(x+1)$, $(y+1)$ and $(z+1)$ is zero, we get

$$z+1=\pm8$$
$$x+1=\pm6$$
$$y+1=\pm4$$

Thus, we have two solutions

$x=5,\ y=3,\ z=7$ and $x=-7,\ y=-5,\ z=-9$.

7. *Prove that, if the coefficients of the quadratic equation $ax^2+bx+c=0$ are odd integers, then the roots of the equation cannot be rational numbers.*

Sol. Let there be a rational root $\left(\dfrac{p}{q}\right)$, where

GCD $(p,\ q)=1$, then

$$\frac{ap^2}{q^2}+\frac{bp}{q}+c=0$$

$$\Rightarrow \quad ap^2+bpq+cq^2=0$$

Now p, q both may be odd or one of p, q should be even. If both p, q are odd, then $ap^2+bpq+cq^2$ is an odd number and cannot be equal to zero. Again, if one of p and q is even, then two of the terms of the left hand side of the equation are even and the third term are odd and again its sum is odd and cannot be equal to zero.

Hence, the above equation cannot have rational roots.

8. *Solve* $\log_2 x+\log_4 y+\log_4 z=2$

$$\log_3 y+\log_9 z+\log_9 x=2$$
$$\log_4 z+\log_{16} x+\log_{16} y=2.$$

Sol. Suppose, $\quad \log_a x=b.$

Then, $\quad x=a^b=(a^n)^{b/n}$

$$\Rightarrow \quad \log_{a^n} x=\frac{b}{n}$$

$$\Rightarrow \quad n\log_{a^n} x = b$$

$$\Rightarrow \quad \log_{a^n} x^n = b = \log_a x$$

So, $\quad \log_2 x = \log_{2^2} x^2 = \log_4 x^2,$

$$\log_3 4 = \log_{3^2} y^2 = \log_9 y^2,$$

and $\quad \log_4 z = \log_{4^2} z^2 = \log_{16} z^2$

So, $\quad \log_2 x + \log_2 y + \log_2 z = 2$

$$\Rightarrow \quad \log_4 x^2yz = 2$$

$$\Rightarrow \quad x^2yz = 16 \qquad \dots (1)$$

Similarly, $\quad xy^2z = 81 \qquad \dots (2)$

and $\quad z^2xy = 256 \qquad \dots (3)$

On multiplying Eqs. (1), (2) and (3), we get

$$(x^2yz)(xy^2z)(xyz^2) = 16 \times 81 \times 256$$

$$\therefore \quad (xyz)^4 = 2^4 \times 3^4 \times 4^4$$

$$\therefore \quad xyz = 24, \text{ as } x,\ y,\ z > 0.$$

On dividing Eqs. (1), (2) and (3) by $xyz = 24$, we get

$$x = \frac{16}{24},\ y = \frac{81}{24},\ z = \frac{256}{24}.$$

$$\Rightarrow \quad x = \frac{2}{3},\ y = \frac{27}{8},\ z = \frac{32}{3}.$$

9. *Find all real numbers satisfying* $x^8 + y^8 = 8xy - 6$.

Sol. We know, $x^8 + y^8 = 8xy - 6$

$$\Rightarrow \quad x^8 + y^8 + 6 = 8xy$$

$$\Rightarrow \quad x^8 + y^8 + 1 + 1 + 1 + 1 + 1 + 1 = 8xy \qquad \dots (1)$$

From, AM $\geq$ GM, we have

$$\frac{x^8 + y^8 + 1 + 1 + 1 + 1 + 1 + 1}{8} \geq (x^8y^8)^{1/8}$$

$$x^8 + y^8 + 6 \geq 8xy \qquad \dots (2)$$

From Eqs. (1) and (2) equality holds when all terms are equal, *i.e.*, $x^8 = y^8 = 1$.

Hence, $x = \pm 1$, $y = \pm 1$ is the solution set.

10. *Find all positive integers x, y, z satisfying*

$$x^{y^z} \cdot y^{z^x} \cdot z^{x^y} = 5xyz.$$

Sol. x, y, z are integers and 5 is a prime number and the given equation is

$$x^{y^z} \cdot y^{z^x} \cdot z^{x^y} = 5xyz$$

On dividing both sides of the equation by xyz, we get

$$x^{y^z - 1} \cdot y^{z^x - 1} \cdot z^{x^y - 1} = 5$$

So, the different possibilities are

$x^{y^z} - 1 = 5$ or $x^{y^z} - 1 = 1$ or $x^{y^z} - 1 = 1$

$y^{z^x} - 1 = 1$ $\quad y^{z^x} - 1 = 5$ $\quad y^{z^x} - 1 = 1$

$z^{x^y} - 1 = 1$ $\quad z^{x^y} - 1 = 1$ $\quad z^{x^y} - 1 = 5$

Taking the first column,

$x = 5$, $y^z - 1 = 1$, $y^z = 2$, $y = 2$ and $x = 1$ and these values are satisfying the other expressions in the first column.

Similarly, from the second column, we get

$y = 5$, $z = 2$ and $x = 1$ and from the third $z = 5$, $x = 2$ and $y = 1$.

Problem 1 *Why do we get two answers on integrating* $\int_{-2}^{2}\frac{dx}{4+x^2}$

i.e.,

Case 1 :

$$\int_{-2}^{2}\frac{dx}{4+x^2}=\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2}=\frac{\pi}{4}$$

Case 2 :

If $I=\int_{-2}^{2}\frac{dx}{4+x^2}$, then by substituting

$$x=\frac{1}{t}\quad\Rightarrow dx=-\frac{1}{t^2}dt$$

$$\therefore\ I=\int_{-1/2}^{1/2}-\frac{1}{t^2\left(4+\frac{1}{t^2}\right)}dt$$

$$=-\int_{-1/2}^{1/2}\frac{dt}{4t^2+1}$$

$$=-\frac{1}{2}[\tan^{-1}(2t)]_{-1/2}^{1/2}$$

$$=-\frac{1}{2}[\tan^{-1}(1)-\tan^{-1}(-1)]$$

$$=-\frac{1}{2}\left[\frac{\pi}{4}+\frac{\pi}{4}\right]=-\frac{\pi}{4}.$$

Prakash, Durgapur

Solution Note that $x=\frac{1}{t}$ is not valid because of discontinuity at $x=0$.

You should always remember,

$$\int_a^b\frac{d}{dx}(f(x))=[f(x)]_a^b,$$

if $f(x)$ is continuous in (a, b).

However, if $f(x)$ is discontinuous in (a,b) at $x=c\in(a,\ b)$,

$$\int_a^b\frac{d}{dx}(f(x))=[f(x)]_a^{c^-}+[f(x)]_{c^+}^{b}$$

$$e.g.,\ \int_{-1}^{1}\frac{d}{dx}\left(\cot^{-1}\frac{1}{x}\right)$$

$$=\left[\cot^{-1}\frac{1}{x}\right]_{-1}^{-0}+\left[\cot^{-1}\frac{1}{x}\right]_{0^+}^{1}$$

$$=\pi-\left(\frac{3\pi}{4}\right)+\frac{\pi}{4}=\frac{\pi}{2}.$$

e.g. $\int_0^{2\pi}\frac{dx}{5-2\cos x}$, the substitution $\tan\frac{x}{2}=t$ is obviously wrong.

e.g., $\int_0^{\pi}\frac{dx}{1+\cos^2 x}=\int_0^{\pi}\frac{\sec^2 x\,dx}{2+\tan^2 x}$; $\tan x=t$ is wrong.

Problem 2 *Describe Squeeze theorem and give some examples.*

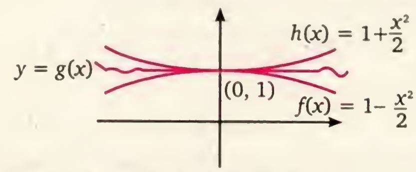
Javed Ansari, Hyderabad

Solution (Sandwich/Squeeze Theorem)

The squeeze principle is used for limit problems where the usual algebraic methods (factorisation or algebraic manipulation etc) are not effective. However, it requires that we are able to "squeeze" our problem in between two other simpler functions whose limits are easily comparable and equal. Use of Squeeze principle requires accurate analysis, indepth algebraic skills and careful use of inequalities.

Statement

If f, g and h are 3 functions such that $f(x)\le g(x)\le h(x)$ for all x in some interval containing the point $x=c$, and if $\lim_{x\to c}f(x)=\lim_{x\to c}h(x)=L$, then $\lim_{x\to c}g(x)=L$.



From the figure, note that $\lim_{x\to 0}g(x)=1$.

The quantity c may be a finite number, $+\infty$ or $-\infty$. Similarly, L may be finite number, $+\infty$ or $-\infty$.

Examples based on Sandwich theorem :

Example 1 $\lim_{x\to\infty}\frac{2-\cos x}{x+3}$

Solution Since, $-1\le\cos x\le 1$

$$\therefore\qquad 1\le 2-\cos x\le 3$$

$$\Rightarrow\qquad \frac{1}{x+3}\le\frac{2-\cos x}{x+3}\le\frac{3}{x+3}$$

$(x+3>0)$ as $x\to\infty$.

$$\Rightarrow\qquad \lim_{x\to\infty}\frac{1}{x+3}\le\lim_{x\to\infty}\frac{2-\cos x}{x+3}\le\lim_{x\to\infty}\frac{3}{x+3}$$

$$\Rightarrow\qquad \lim_{x\to\infty}\frac{2-\cos x}{x+3}=0$$

Example 2 $\lim_{x\to 0}x^3\cos\frac{2}{x}$.

Solution Since, $-1\le\cos\frac{2}{x}\le 1$,

$$\therefore\qquad -x^3\le x^3\cos\frac{2}{x}\le x^3\qquad\text{for } x>0$$

$$\Rightarrow\qquad -x^3\le x^3\cos\frac{2}{x}\le x^3\qquad\text{for } x<0$$

In both the cases limit is zero.

$$\therefore\qquad \lim_{x\to 0}x^3\cos\frac{2}{x}=0$$

Example 3 *Evaluate*

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+\ldots+\frac{n}{n^2+n}\right)$$

Solution Let $f(n)=\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+\ldots+\frac{n}{n^2+n}$

Note that $f(n)$ has n terms which are decreasing.

Solution Since, the slope of linear function is constant.

$$\therefore \quad \frac{f(6)-f(2)}{6-2}=\frac{f(12)-f(2)}{12-2}$$

$$\Rightarrow \quad f(12)-f(2)=30.$$

Problem 12 *There are three teachers and six students. Number of ways in which they can be seated in a line so that between any two teachers there are exactly 2 students, is*

(*a*) $3\cdot 3!\cdot 6!$ (*b*) $2\cdot 6!$

(*c*) $2\cdot 3!\cdot 6!$ (*d*) $3\cdot 6!$

Priyanaka Sharma, *Noida*

Solution Six students can be seated in 6! ways

$$|\,S_1S_2\,|\,S_3S_4\,|\,S_5S_6$$

Corresponding to each of these six ways, teachers can be seated in two ways and can be arranged in 3! ways.

$$S_1S_2|\,S_3S_4\,|\,S_5S_6|$$

$\therefore$ Total number of ways $= 2\cdot 3!\cdot 6!$.

Hence, (c) is the correct answer

Problem 13 *The average of the numbers* $n \sin n°$ *for* $n = 2, 4, 6, \ldots, 180$, *is*

(*a*) 1 (*b*) $\cot 1°$

(*c*) $\tan 1°$ (*d*) $\frac{1}{2}$

Ratinder Sodhi, *Chandigarh*

Solution $S = 2\sin 2°+4\sin 4°+\ldots\ldots+180\sin 180°$

$S = 2[\sin 2°+2\sin 4°+3\sin 6°+\ldots\ldots+89\sin 178°]$

$S = 2[89\sin 178°+88\sin 176°+\ldots\ldots+1\sin 2°]$

....

$2S = 2[90(\sin 2°+\sin 4°+\sin 6°+\ldots\ldots+\sin 178°)]$

$$S = 90\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\frac{\theta}{2}}\sin\left(\frac{(n+1)\theta}{2}\right)=\frac{90\sin(89°)}{\sin 1°}\cdot\sin 90°$$

$$\Rightarrow \quad S = 90\cot 1°$$

$$\text{Average value} = \frac{90\cot 1°}{90} = \cot 1°.$$

Hence, (b) is the correct answer.

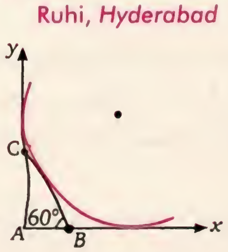
Problem 14 *A circle with centre O is tangent to the coordinate axes and to the hypotenuse of the triangle ABC as shown below, where* $AB = 1$. *To the nearest hundredth, the radius of the circle, will be*

(*a*) 2.37 (*b*) 2.24

(*c*) 2.18 (*d*) 2.41

Ruhi, *Hyderabad*

Solution $r_1 = s\tan\frac{A}{2}$



Therefore, $\frac{A}{2}=45°$; $s=\frac{3+\sqrt{3}}{2}$

$$\therefore r_1 = \frac{3+\sqrt{3}}{2}=\frac{4.732}{2}$$

$= 2.366 = 2.37.$

Hence, (a) is the correct answer.

Problem 15

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 3 & & 5 & & & \\ & & 7 & & 9 & & 11 & & \\ & 13 & & 15 & & 17 & & 19 & \\ 21 & & 23 & & 25 & & 27 & & 29 \end{array}$$

Consecutive odd numbers are arranged in rows as shown. If the rows are continued in the same pattern, then the middle number of the row 51 will be

(*a*) 2601 (*b*) 2500

(*c*) 2704 (*d*) 2401

Rakesh Garg, *Faridabad*

Solution Only odd rows have a middle number, and the middle number in rows 1, 3, 5 are 1, 9, 25 that is 1^2, 3^2, 5^2. Thus, the middle number in row 51 will be $51^2 = 2601$.

Problem 16 *Number of seven digit whole numbers in which only 2 and 3 are present as digits, if no two 2's are consecutive in any number, is*

(*a*) 26 (*b*) 33

(*c*) 32 (*d*) 53

Neha Dutt, *Panchkula*

Solution

Exactly one 2's $= {}^7C_1$ _×_×_×_×_×_×_(Six 3's)

Exactly two 2's $= {}^6C_2$ _×_×_×_×_×_(Five 3' s)

Exactly three 2's $= {}^5C_3$ _×_×_×_×_ (Four 3' s)

Exactly four 2's $= {}^4C_4 = 1$ _×_×_×_ (Three 3's)

Total $= 7 + 15 + 10 + 1 = 33$.

Hence, (b) is the correct answer.

Problem 17 *Let ABCDE be a regular pentagon with all sides equal to 4. Which of the following is a correct solution for the length AC ?*

(*i*) $2\cos(18°)$

(*ii*) $2\sec(72°)$

(*iii*) $\sqrt{32-32\cos(108°)}$

(*a*) Only (*i*) and (*ii*) are correct

(*b*) Only (*ii*) and (*iii*) are correct

(*c*) Only (*iii*) and (*i*) are correct

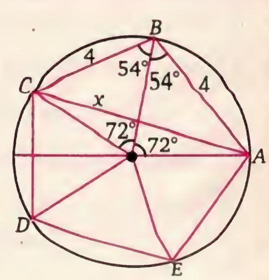
(*d*) All are correct

Rashmi Thakur, *Lucknow*

Solution Using cosine rule,

$$x^2 = 16+16-2\cdot 4\cdot 4\cos(108°)$$

$$= 32(1-\cos 108°)$$



$\Rightarrow x = 8\sin 54°$

$\therefore x = 8\cos 36° = 2(\sqrt{5}+1)$.

Now, verify (i) and (ii) also.

$\Rightarrow$ All options are correct.

Hence, (d) is the correct answer.

The $(n+1)^{th}$ Dimension

1. Let $f(x)=x^2-1$ and $g(x)=\begin{cases}[|f(|x|)]+|[f(x)]|; & x\in(-1,0)\cup(0,1)\\ 1; & \text{otherwise}\end{cases}$

   Then, find the range of ln ($[|g(x)|]$), where [ ] denotes the greatest integer function.

2. For what values of 'a' the point of local minima of $f(x)=x^3-3ax^2+3(a^2-1)x+1$ is less than 4 and point of local maxima is greater than -2.

3. Two chords PQ and PR are drawn from a point P of the parabola $y^2=4ax$ and are at right angles. Prove that for every fixed position of point P, there exists a fixed point T on QR. Also find the locus of point T.

4. Let z_1, z_2, z_3 be the complex numbers such that $|z_1-1|=|z_2-1|=|z_3-1|$ and $\arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)=\dfrac{\pi}{6}$, then prove that

   $z_2(z_2-1)-z_3(z_2+1)+(z_3+1)(z_3-1)+2=0.$

5. If complex number z lies on the curve $|z-(-1+i)|=1$, then find the locus of the complex number $w=\dfrac{z+i}{1-i}, i=\sqrt{-1}$.

6. Evaluate $\displaystyle\int\frac{dx}{\tan x+\cot x+\sec x+\operatorname{cosec} x}$.

7. Let G be the centroid of the triangle ABC in which the angle at C is obtuse and let AD and CF be medians from A and C respectively onto the sides BC and AB. If the four points B, D, G and F are concyclic, show that $AC-\sqrt{2}\,BC>0$.

8. If $x^5-x^3+x=a$, then prove that $x^6\geq 2a-1$.

9. Solve $y(2xy+1)dx+x(1+2xy+x^2y^2)dy=0$.

10. If roots of the quadratic equation $x^2-(2n+18)x-n-11=0$, $n\in$ set of integers, are rational, then find the value (s) of n.

11. From a point $(1,1,21)$, a ball is dropped onto the plane $x+y+z=3$, where x, y-plane is horizontal and z-axis is along the vertical. Find the coordinates of the point where the ball hits the plane the second time. (Use $s=ut-1/2gt^2$ and $g=10$ m/s^2)

12. A boat is rowed with a velocity u directly across a stream of width a. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the path of the boat and the distance downstream to the point where it lands.

13. Let f (x) be a function which satisfies the equation $f(xy) = f(x) + f(y)$ for all $x > 0, y > 0$ such that $f'(1) = 2$. Find the area of the region bounded by the curves $y = f(x)$, $y = |x^3 - 6x^2 + 11x - 6|$ and $x = 0$.

14. Let PQ be the common chord of the parabola $y^2 = 4ax$ and the circle touching the parabola at P and Q. Tangent at P and the chord PQ are equally inclined to the axes of the parabola. Prove that the locus of middle point of PQ is another parabola whose latus rectum is one fifth of the given parabola.

From any point P on the curve $b^4 x + 2a^2y^2 = 0$, a pair of tangents PQ and PR are drawn on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Prove that QR touches a fixed parabola. Also find the equation of this parabola.

15. If $1 \cdot 0! + 3 \cdot 1! + 7 \cdot 2! + 13 \cdot 3! + 21 \cdot 4! + \ldots$ upto $(n+1)$ terms $= 4000.(4000!)$, then find the value of n.

Explanations

1. $|f(|x|)| = 1 - x^2, x \in (-1, 0) \cup (0, 1)$

$\Rightarrow \quad [|f(|x|)|] = 0, x \in (-1, 0) \cup (0, 1)$

Also $\quad [f(x)] = -1, x \in (-1, 0) \cup (0, 1)$

$\Rightarrow \quad |[f(x)]| = 1, x \in (-1, 0) \cup (0, 1)$

$\therefore \quad g(x) = 1, x \in R$

$\Rightarrow$ Range of $\ln([|g(x)|]) = \{0\}$.

2. $f'(x) = 3(x^2 - 2ax + a^2 - 1)$

Clearly, roots of the equation $f'(x) = 0$ must be distinct and lie in the interval $(-2, 4)$.

$\therefore \quad D > 0 \Rightarrow a \in R \quad \ldots (1)$

$f'(-2) > 2 \Rightarrow a^2 + 4a + 3 > 0$

$\Rightarrow \quad a < -3$ or $a > -1 \quad \ldots (2)$

$f'(4) > 0 \Rightarrow a^2 - 8a + 15 > 0$

$\Rightarrow \quad a > 5$ or $a < 3 \quad \ldots (3)$

and $-2 < -\frac{B}{2A} < 4 \Rightarrow -2 < a < 4 \quad \ldots (4)$

From Eqs. (1), (2), (3), we have $-1 < a < 3$

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$f'(x) = 3(x - (a-1))(x - (a+1))$

Clearly, $-2 < a + 1 < 4$ and $-2 < a - 1 < 4$

$\Rightarrow -1 < a < 3$.

3. Let $P \equiv (at^2, 2at)$, $Q \equiv (at_1^2, 2at_1)$, $R \equiv (at_2^2, 2at_2)$

Slope of line $PQ = \frac{2}{t + t_1}$

Slope of line $PR = \frac{2}{t + t_2}$

Since PQ and PR are perpendicular

$\therefore \quad \frac{2}{t + t_1} \cdot \frac{2}{t + t_2} = -1$

$\Rightarrow \quad (t + t_1)(t + t_2) = -4$

$\Rightarrow \quad t_1t_2 = -4 - t^2 - t(t_1 + t_2) \quad \ldots (1)$

Also equation of line QR is

$$y(t_1 + t_2) = 2(x + at_1t_2) \quad \ldots (2)$$

From Eqs. (1) and (2), we have

$y(t_1 + t_2) = 2[x + a\{-4 - t^2 - t(t_1 + t_2)\}]$

$= 2(x - 4a - at^2) - 2at(t_1 + t_2)$

$\Rightarrow y + 2at - \frac{2}{t_1 + t_2} = (x - 4a - at^2) = 0$

$L_1 + \lambda L_2$

$\Rightarrow y + 2at - (x - 4a - at^2) = 0$

$\Rightarrow$ This family passes through the point of intersection of $y + 2at = 0$ and $x - 4a - at^2 = 0$

$\Rightarrow \beta = -2at$ and $\alpha = 4a + at^2$, where $T \equiv (\alpha, \beta)$

Eliminating t, we get

$$\alpha = 4a + \frac{\beta^2}{4a}$$

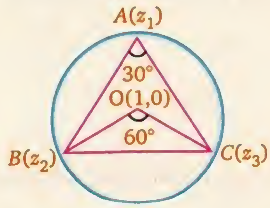
$\therefore$ The required equation of locus is $4ax = 16a^2 + y^2 = 0$

$\Rightarrow y^2 - 4ax + 16a^2 = 0$

which is a parabola.

4. Clearly z_1, z_2 and z_3 lies on a circle centred at $(1, 0)$.

Also, $\angle BAC = 30°$



$\Rightarrow \Delta$ BOC is equilateral

$\Rightarrow z_2^2 + z_3^2 + 1 = z_2 + z_3 + z_2z_3$

$\Rightarrow z_2(z_2 - 1) - z_3(z_2 + 1) + (z_3 + 1)(z_3 - 1) + 2 = 0.$

5. $|z - (-1 + i)| = 1$

$\Rightarrow \quad |z + 1 - i| = 1$

Also $\quad w = \frac{z + i}{1 - i}$

$$\Rightarrow (1-i)w = z+i \Rightarrow (1-i)w - i = z$$

$$\Rightarrow \quad |(1-i)w-i+1-i| = |z+1-i|$$

$$\Rightarrow |1-i|\left|w+\frac{1-2i}{1-i}\right| = 1 \Rightarrow \left|w+\frac{(1-2i)(1+i)}{(1+i)(1-i)}\right| = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \left|w+\frac{3-i}{2}\right| = \frac{1}{\sqrt{2}} \Rightarrow \left|w-\frac{-3+i}{2}\right| = \frac{1}{\sqrt{2}}$$

Hence, locus of w is a circle centered at $\left(-\frac{3}{2}, \frac{1}{2}\right)$ and having radius $\frac{1}{\sqrt{2}}$.

6. Given integral can be rewritten as $I = \int \frac{\sin x \cos x\, dx}{1+\sin x+\cos x}$

$$= \int \frac{\sin x\, dx}{\sec x + \tan x + 1} = \int \frac{\sin x(1+\tan x - \sec x)}{2\tan x}\, dx$$

$$= \int \frac{1}{2}\cos x(1+\tan x-\sec x)dx$$

$$= \int \frac{1}{2}(\cos x + \sin x - 1)dx$$

$$= \frac{1}{2}(\sin x - \cos x - x) + C$$

7. From Δs ABD and ADC, we have

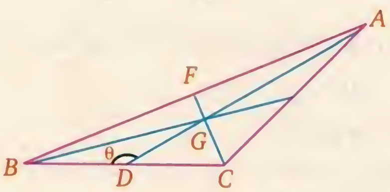
$AB^2 = AD^2 + BD^2 - 2AD \cdot BD \cdot \cos\theta$, and ...(1)

$AC^2 = AD^2 + DC^2 + 2AD \cdot DC \cdot \cos\theta$ ...(2)

On adding Eqs. (1) and (2), we get

$$c^2 + b^2 = 2AD^2 + \frac{a^2}{2}$$

$$\Rightarrow \quad AD^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4} \quad \text{...(3)}$$



Similarly, other medians are given by

$$BE^2 = \frac{c^2+a^2}{2} - \frac{b^2}{4}, \text{ and} \quad \text{...(4)}$$

$$CF^2 = \frac{a^2+b^2}{2} - \frac{c^2}{4} \quad \text{...(5)}$$

Since B, D, G, F are concyclic, we have

$AF \cdot AB = AG \cdot AD$

But, $AG = \frac{2}{3}AD$

Hence, $\frac{1}{2}c^2 = \frac{2}{3}AD^2 = \frac{1}{3}\left(b^2 + c^2 - \frac{a^2}{2}\right)$

$$\Rightarrow \quad \frac{3}{2}c^2 = b^2 + c^2 - \frac{a^2}{2}$$

$$\Rightarrow \quad b^2 = \frac{1}{2}(c^2 + a^2)$$

Also, we have, $b^2 = c^2 + a^2 - 2ca\cos B = 2b^2 - 2ca\cos B$

$\Rightarrow b^2 = 2ca\cos B$

Also, $a = c\cos B + b\cos C < c\cos B$ (since $\angle C > 90°$)

Hence, $2a^2 < 2c\cos B = b^2$

$$\Rightarrow \quad \frac{b}{a} > \sqrt{2} \Rightarrow \frac{AC}{BC} > \sqrt{2}.$$

8. Given that $x(x^4 - x^2 + 1) = \frac{x(x^6+1)}{x^2+1}$

Suppose that $x > 0$, then $a > 0$

$$\Rightarrow x^6 + 1 = a\left(\frac{x^2+1}{x}\right) = a\left(x + \frac{1}{x}\right) \geq 2a$$

$$\Rightarrow \quad x^6 \geq 2a - 1 \quad \text{...(1)}$$

If $\quad x \leq 0 \Rightarrow a \leq 0 \Rightarrow 2a - 1 < 0 \leq x^6$ ...(2)

From Eqs. (1) and (2), we get $x^6 \leq 2a - 1$.

9. The given differential equation is $y(2xy+1)dx + x(1+2xy+x^2y^2)dy = 0$.

$$\Rightarrow y(2xy+1)dx + x(1+2xy)dy + x^3y^2dy = 0$$

$$\Rightarrow (2xy+1)(y\,dx + x\,dy) + x^3y^2dy = 0$$

$$\Rightarrow (2xy+1)d(xy) = -x^3y^3\frac{1}{y}dy$$

$$\Rightarrow \quad \frac{2xy+1}{(xy)^3}d(xy) + \frac{dy}{y} = 0$$

On integrating, we get

$$-\frac{2}{xy} - \frac{1}{2(xy)^2} + \ln y = C$$

$$\Rightarrow \quad \ln y = c + \frac{2}{xy} + \frac{1}{2(xy)^2}.$$

10. Here, discriminant of the given equation must be a perfect square.

i.e., $4[(n+9)^2 + n + 11]$ must be perfect square.

$\Rightarrow n^2 + 19n + 92$ must be perfect square.

$$\Rightarrow \quad n^2 + 19n + 92 = m^2$$

$$\Rightarrow \quad n = \frac{-19 \pm \sqrt{4m^2 - 7}}{2}$$

$\Rightarrow \quad 4m^2 - 7$ is a perfect square.

$$\Rightarrow \quad 4m^2 - 7 = p^2 \Rightarrow 4m^2 - p^2 = 7$$

$$\Rightarrow \quad (2m+p)(2m-p) = 7$$

$\Rightarrow$ either $\quad 2m + p = \pm 1, 2m - p = \pm 7$

or $\quad 2m + p = \pm 7, 2m - p = \pm 1$

$$\Rightarrow \quad 2m = \pm 4 \Rightarrow m = \pm 2$$

$$\Rightarrow \quad n^2 + 19n + 92 = 4$$

$$\Rightarrow (n+8)(n+11) = 0 \Rightarrow n = -8 \text{ or } -11.$$

11. Since, it falls along the vertical, the *xy*–coordinates of the ball will not change before it strikes the plane.

$\Rightarrow$ If Q be the point where the ball meets the plane for the 1st time, then

$Q \equiv (1, 1, 1)$.

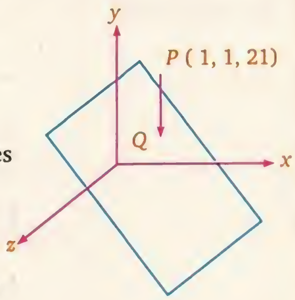
Speed of the ball just before striking the plane is

$$\sqrt{2 \times 10 \times 20} = 20 \text{ m/s}.$$

Now let θ be the angle between PQ and normal to the plane.

$$\therefore \cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \cos 2\theta = -\frac{1}{3}, \sin 2\theta = \frac{2\sqrt{2}}{3}$$

Now, component of velocity in the direction of *z*-axis after it strikes the plane



$$= -20 \sin\left(2\theta - \frac{\pi}{2}\right) = -\frac{20}{3} \text{ m/s}$$

Hence, in t time the *z*–coordinate of ball becomes

$$1 - \frac{20}{3}t - \frac{1}{2} \times 10t^2 = 1 - \frac{20}{3}t - 5t^2$$

The component of velocity in *xy*–plane is

$$20\cos\left(2\theta - \frac{\pi}{2}\right) = 20 \sin 2\theta = \frac{20 \times 2\sqrt{2}}{3} = \frac{40\sqrt{2}}{3}$$

Using symmetry, the component along the *x*–axis $= \dfrac{40}{3}$

and the component along the *y*–axis $= \dfrac{40}{3}$

Hence, x and y coordinates of the ball after time $t = 1 + \dfrac{40}{3}t$

$\Rightarrow$ after time t the coordinate of the ball will become

$$\left(1 + \frac{40}{3}t,\ 1 + \frac{40}{3}t,\ 1 - \frac{20}{3}t - 5t^2\right)$$

It lies on the plane $\dfrac{80}{3}t - \dfrac{20}{3}t - 5t^2 = 0$

$$\Rightarrow \quad 20t - 5t^2 = 0$$

$$\Rightarrow \quad t = 4.$$

$\Rightarrow$ coordinate of the point where the ball strikes the plane in second time

$$= \left[\frac{163}{3}, \frac{163}{3}, -\frac{317}{3}\right].$$

12. Let us take the origin at the point from where the boat starts.

At any time t after its start from O, let the boat be at $P(x, y)$, so that

$$\frac{dx}{dt} = \text{velocity of the current}$$

$$= ky(a - y)$$

and $\dfrac{dy}{dt}$ = velocity with which the boat is being rowed

$$= u$$

Now, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{u}{ky(a-y)} \qquad \ldots(1)$$

This gives the direction of the resultant velocity of the boat which is also the direction of the tangent to the path of the boat.

Now, from Eq. (1)

$$y(a-y)dy = \frac{u}{k}dx$$

$$\Rightarrow \quad a\frac{y^2}{2} - \frac{y^3}{3} = \frac{u}{k}x + C$$

Since, $y = 0$, when $x = 0 \Rightarrow C = 0$

Hence, the equation to the path of the boat is

$$x = \frac{k}{6u}y^2(3a-2y)$$

On putting $y = a$, we get the distance AB, downsteam where that boat lands

$$= \frac{ka^2}{6u}$$

13. Take $x = y = 1 \Rightarrow f(1) = 0$

Now $0 = f\left(x \cdot \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$

$$\Rightarrow \quad f\left(\frac{1}{x}\right) = -f(x)$$

$$\therefore \quad f\left(\frac{x}{y}\right) = f(x) + f\left(\frac{1}{y}\right)$$

$$= f(x) - f(y)$$

Now, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{1}{h}f\left(\frac{x+h}{h}\right)$$

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}\cdot x}$$

$$= \frac{f'(1)}{x} = \frac{2}{x}$$

$\Rightarrow \quad f(x) = 2\ln x + C \Rightarrow C = 0 \qquad$ (since $f(1) = 0$)

$\Rightarrow \quad f(x) = 2\ln x$

Required area

$$= \int_0^1 (x^3 - 6x^2 + 11x - 6)\,dx + \int_{-\infty}^0 e^{y/2}\,dy$$

$$= \frac{7}{4}\text{ sq unit}$$

14. Let $P \equiv (at_1{}^2, 2at_1); Q \equiv (at_2{}^2, 2at_2)$

Slope of tangent at $P = \frac{1}{t_1}$

Slope of $PQ = \frac{2}{t_1+t_2}$

As per the given condition,

$$\frac{1}{t_1} = -\frac{2}{t_1+t_2}$$

$\Rightarrow \quad 3t_1 + t_2 = 0 \Rightarrow t_2 = -3t_1 \qquad \ldots(1)$

Let (h, k) be the middle point of PQ.

$\Rightarrow \quad h = \frac{1}{2}a(t_1^2 + t_2^2)$ and $k = a(t_1 + t_2) \Rightarrow k = -2at_1$

or $\quad k^2 = 4a^2t_1{}^2$

and $\quad 2h = a(t_1{}^2 + at_1{}^2) = 10\,at_1{}^2 \qquad$ (from Eq. (1))

Eliminating 't_1', we get

$\frac{k^2}{2h} = \frac{4a}{10} \Rightarrow 5k^2 = 4ah \Rightarrow$ locus is $5y^2 = 4ax$.

Any point on the curve $b^4x + 2a^2y^2 = 0$, is $\left(-\frac{2a^2t^2}{b^4}, t\right)$

Equation of QR is

$$\frac{x}{a^2}\left(-\frac{2a^2t^2}{b^4}\right) - \frac{yt}{b^2} = 1$$

$$\Rightarrow \quad -\frac{2xt^2}{b^4} - \frac{yt}{b^2} = 1$$

$$\Rightarrow \quad -2xt^2 - ytb^2 = b^4 \qquad \ldots(1)$$

Now we have to prove that Eq. (1) touches a fixed parabola.

From Eq. (1), we have $b^2\,ty = 2\,xt^2 - b^4$

$$\Rightarrow \quad y = \frac{-2xt^2}{b^2t} - \frac{b^4}{b^2t}$$

$$\Rightarrow \quad y = -\frac{2t}{b^2}x - \frac{b^2}{t}$$

$$\Rightarrow \quad y = \left(-\frac{2t}{b^2}\right)x + \frac{2}{(-2t/b^2)} \qquad \ldots(2)$$

Clearly Eq. (2) is of the form $y = mx + \frac{2}{m}$, which always touches the parabola $y^2 = 8x$ for all real values of t.

15. Clearly $t_r = (r^2 + r + 1)\cdot r!;\ r = 0, 1, 2, \ldots n$

i.e. $\quad t_r = [(r+1)^2 - r]\,r!$

$$= (r+1)(r+1)r! - r\cdot r!$$
$$= (r+1)\cdot(r+1)! - (r+1-1)\cdot r!$$
$$= (r+2-1)(r+1)! - (r+1)!r! + r!$$
$$= (r+2)! - (r+1)! - (r+1)! + r!$$
$$= (r+2)! - 2(r+1)! + r!$$

$\therefore \quad t_0 = 2! - 2\cdot 1! + 0!$

$t_1 = 3! - 2\cdot 2! + 1!$

$t_2 = 4! - 2\cdot 3! + 2!$

$t_3 = 5! - 2\cdot 4! + 3!$

....

....

$t_{n-1} = (n-1)! - 2n! + (n-1)!$

$t_n = (n+2)! - 2(n+1)! + n!$

$$\Rightarrow \quad \sum_{r=0}^{n} t_r = (n+2)! + (n+1)! - 2(n+1)! - 2\cdot 1! + 0! + 1!$$

$$= (n+2)! - (n+1)! = (n+1)!(n+2-1)$$

$$= (n+1)(n+1)!$$

Given $\quad (n+1)(n+1)! = 4000\,(4000)!$

$\Rightarrow \quad n + 1 = 4000 \Rightarrow n = 3999.$